Dissipative lattice model with exact traveling discrete kink-soliton solutions: Discrete breather generation and reaction diffusion regime

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We introduce a nonlinear Klein-Gordon lattice model with specific double-well on-site potential, additional constant external force and dissipation terms, which admits exact discrete kink or traveling wave front solutions. In the nondissipative or conservative regime, our numerical simulations show that narrow kinks can propagate freely, and reveal that static or moving discrete breathers, with a finite but long lifetime, can emerge from kink-antikink collisions. In the general dissipative regime, the lifetime of these breathers depends on the importance of the dissipative effects. In the overdamped or diffusive regime, the general equation of motion reduces to a discrete reaction diffusion equation; our simulations show that, for a given potential shape, discrete wave fronts can travel without experiencing any propagation failure but their collisions are inelastic.

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I. INTRODUCTION

In recent years, the dynamics of kinks in nondissipative (Klein-Gordon) systems (for a recent review see Braun and Kivshar [1]) or traveling wave fronts in strongly dissipative or reaction diffusion systems has attracted considerable attention. It has become clear that continuous propagation equations and reaction diffusion equations provide an inadequate description of the behavior of weakly coupled lattices where the interplay between nonlinearity and spatial discreteness can lead to novel effects not present in the continuum models. For example, in nondissipative (or weakly) lattices such as: ferromagnetic chains [2], hydrogen bonded chains [3], or chain of base pairs in DNA [4], kink solitons or domain walls, whose width is of the order of few lattice spacings, may pin on the lattice owing to discreteness effects. On the other hand, in strongly dissipative lattices of coupled excitatory cells, which are used as models in neurophysiology [5,6] and cardiophysiology [7] in order to describe wave propagation in nerve cells, wave propagation failure, which also originates from lattice discreteness effects [8] is an important phenomenon which may often lead to breakdown of these systems with potentially fatal consequences.

In order to gain understanding of wave motions in discrete systems, for which exact results are scarce even in one dimension, it is desirable to investigate lattice models with exact solutions. In this regard, Schmidt [9] pointed out that if the double-well on-site potential of the φ4 lattice model is suitably modified, the single kink soliton becomes an exact solution to the discrete model; later, this model was shown to be integrable in the static limit and admit exact static solutions into the form of generally unpinned soliton lattices (Jensen et al. [10]). On the other hand, Bressloff and Rowlands [11], using an approach which presents some similarities with Jensen et al. [10], have recently shown that it is possible to construct exact traveling wave solutions of a discrete reaction diffusion equation, describing a system of coupled bistable elements, if the form of the bistable potential is adequately chosen. In fact, their potential corresponds precisely to Schmidt’s potential. Very recently, the general problem of finding kink or pulse shaped traveling waves solutions was considered by Flach and coworkers [12]. As have the above-mentioned authors, they have approached the traveling wave existence problem from the inverse side, that is, they have shown that for a given wave problem, corresponding equations of motion can be generated, so that these equations yield the chosen wave profile as a solution. They have studied conservative lattice models and dissipative (reaction diffusion like) models, separately. One might therefore wonder if it is possible, in a similar way, to construct a general discrete model including both inertia and dissipation. Actually, generalizing Schmidt’s approach, we show in this paper, that a lattice model with on-site double-well potential, with additional external force and dissipation terms, can also admit exact kink or traveling wave front solutions in any regime: nondissipative or dissipative.

The paper is organized as follows. First, we present our specific lattice model and show analytically that it can admit exact discrete kink solutions if the double-well on-site potential is adequately chosen. Then, in Sec. III, we study numerically the propagation and collisions of discrete kinks and antikinks, in the nondissipative and dissipative regimes. In Sec. IV, we consider the reaction diffusion regime and investigate numerically the properties of traveling discrete wavefronts solutions. Section V is devoted to concluding remarks.

II. MODEL AND EQUATION OF MOTION

We consider a chain of harmonically coupled particles of mass m, lying in a double well on site potential U. This system is modeled by the general discrete equation of motion
\[ m \frac{d^2 u_n}{dt^2} + \Gamma \frac{du_n}{dt} = k[u_{n+1} - 2u_n + u_{n-1}] - F - \frac{dU(u_n)}{du_n}, \]  
(2.1)

where \( u_n \) is the \( n \)th particle displacement, \( \Gamma \) is a dissipative coefficient, and \( k \) is a coupling term. \( F \) is a constant term which may be an external force and the energy gain due to \( F \) is compensated by the loss mechanism. We assume that a traveling wave front solution of (2.1) has the following kink shape:

\[ u_n(t) = u_0 \tanh(\omega t - \kappa n a). \]  
(2.2)

Here, \( u_0 \) is the amplitude, \( a \) the lattice spacing, and \( \omega \) and \( \kappa \) are two constants such that the ratio \( c = \omega / \kappa \) represents the velocity of the front. Following Schmidt [9], we construct a potential, including the external potential, which has the form:

\[ V(u_n) = Fu_n + U(u_n) \]
\[ = Fu_n - A \frac{u_n^2}{2} + B_0 u_n^4 + B \frac{u_n^8}{8} + C_0 \frac{u_n^6}{6} + C u_n^4 + D_0 u_n^7 + D \frac{u_n^9}{8} + \ldots, \]  
(2.3)

such that the expression (2.2) becomes an exact discrete solution of (2.1). Here, \( A, B, B_0, C, C_0, D_0, D \), represent constant coefficients. The detailed calculations are presented in the Appendix. We obtain the new equation of motion

\[ \frac{m}{k} \frac{d^2 \psi_n}{dt^2} + \Gamma \frac{d\psi_n}{dt} = -\frac{dV(\psi_n)}{d\psi_n} + [\psi_{n+1} - 2\psi_n + \psi_{n-1}], \]  
(2.4)

where we have introduced the dimensionless variable \( \psi_n = \tau (u_n/u_0) \), with \( \tau = \tanh(\alpha a) \). Here \( V(\psi_n) \) is given by

\[ V(\psi_n) = \alpha \psi_n + \beta \psi_n^3 + \gamma \psi_n^5 + \delta \psi_n^7 + \xi \ln(1 - \psi_n^2), \]  
(2.5)

with \( \alpha = -\Gamma \omega \tau / k \), \( \beta = (m \omega^2 / k - 1) \), \( \gamma = \Gamma \omega \tau (k \tau) \), \( \delta = -m \omega^2 / (2k \tau^2) \), and \( \xi = -2(\tau^2 - 1) \). The propagation velocity of the front is given by

\[ c = \frac{\omega}{\kappa} = \frac{\alpha \omega}{\arctanh(\tau)}. \]  
(2.6)

The general discrete equation of motion (2.4) with potential (2.5) and solution

\[ \psi_n = \tau \tanh(\omega t - \kappa n a), \]  
(2.7)

is valid for any value of \( F \) and \( \Gamma \). At this stage, three different cases can be considered.

(i) When \( m \neq 0 \) and \( \Gamma = 0 \), the coefficients of \( \psi_n \) and \( \psi_n^3 \) in (2.5) become zero, the symmetry breaking due to \( F \) disappears and the potential becomes symmetric with two degenerate minima. One recovers the conservative system studied by Schmidt [9]; the total energy is constant and to each \( (\omega, \kappa) \) combination corresponds a different well shape and velocity.

(ii) When \( m \neq 0 \), \( \Gamma \neq 0 \), this is the general case where both inertial and dissipative effects play a role. The value of \( \omega \) is imposed [see Eq. (A7)] by the relation \( F = -\omega_0 \Gamma \). It turns out that the potential well shape and velocity both depend uniquely on the choice of \( \kappa \), that is \( \tau \).

(iii) When \( \Gamma = 0 \) and \( m = 0 \), the lattice dynamics becomes overdamped or diffusive. In (2.5), \( \beta \), the coefficient of \( \psi_n^5 \) reduces to \(-1 \) and \( \delta \), the coefficient of \( \psi_n^7 \) becomes zero; the potential is asymmetric with two nondegenerate minima. This regime corresponds to the model studied by Bressloff and Rowlands [11].

In the following we consider these three cases successively.

### III. NONDISSIPATIVE AND DISSIPATIVE REGIMES

#### A. Nondissipative regime

In this case (\( \Gamma = 0 \)), the potential, represented in Fig. 1, becomes

\[ V(\psi_n) = \left( m \frac{\omega^2}{k} - 1 \right) \psi_n^2 - m \frac{\omega^2}{2k \tau^2} \psi_n^4 - 2(\tau^2 - 1) \ln(1 - \psi_n^2). \]

(3.1)

The barrier height is given by (3.2),

\[ E_0 = \left| \tau^2 \left( 1 - \frac{\omega^2}{2\omega_0^2} \right) - (1 - \tau^2) \ln(1 - \tau^2) \right|. \]

(3.2)

with \( \omega_0^2 = k/m \). The equation of motion (2.4) is written as

\[ \frac{d^2 \psi_n}{dt^2} = \omega_0^2 [\psi_{n+1} - 2\psi_n + \psi_{n-1}] \]
\[ + \omega_0^2 \frac{2\omega^2}{\omega_0^2 \tau^2} \psi_n^3 - \psi_n^2 + 2\psi_n - 2(1 - \tau^2) \frac{\psi_n}{1 - \psi_n^2}. \]

(3.3)
respectively, $k$ cells. The parameters of the lattice and the potential are, width of five lattice spacings traveling in a lattice of 2048 cells. The parameters of the lattice and the potential are, respectively, $k=0.1$ and $m=1$, $\omega=0.0493$ and $\tau=0.761$. Under these conditions, the velocity of the kink is $c=0.023$ cell s$^{-1}$. Note that no radiation effects are observed which confirms that (2.7) is an exact solution.

Then we have studied the possible generation of nonlinear localized modes or discrete breathers via kink ($K$) and antikink ($\bar{K}$) collisions. This investigation was motivated by the interesting properties of discrete breathers: these nontopological excitations can exist in a large variety of nonlinear lattices and their existence is associated with a localization of energy (for recent reviews see [13,14]). Specifically, we have studied numerically a $K-\bar{K}$ collision where the two entities travel at velocity $c=0.023$ cell s$^{-1}$ and $-c$, respectively. As shown in Fig. 3(a), a discrete stationary breather and a small amplitude radiation background emerge from the weakly inelastic $K-\bar{K}$ collision. In Fig. 3(b), we have represented this breather at three different times, $t=500$ s, $t=510$ s, and $t=520$ s, respectively, such that we can observe its behavior over one period of oscillation $T_B=20$ s. The oscillations of the central particle are symmetric with an amplitude $\Delta v_{\text{max}}=1.36$ (see Fig. 1).

$$1-2(\omega^2/\omega_0^2)-(1+\tau^2)(1-\tau^2).$$

As a result, this small radiation can propagate. Nevertheless, in spite of these weak radiation losses, this discrete breather has an important lifetime and presents a physical interest. We have also investigated the collision of a static kink ($c=0$) with a kink moving at velocity $c=0.023$ cell s$^{-1}$. Such a collision results in a discrete breather with the same properties as the static one, but moving at mean velocity $c_b=0.013$ cell s$^{-1}$.

**B. General dissipative regime**

In the general regime, where both inertial and dissipative terms are present, we have analyzed numerically a $K-\bar{K}$ collision. As pointed out in Sec. II, the potential well shape and velocity both depend on the choice of $\tau$. In this case, as it should be expected, we have observed that the number $N$ of breathing oscillations depends on $\Gamma/m$. For example, we have the following results: $N=27$ for $\Gamma/m=0.001, N=8$ for $\Gamma/m=0.01$, and $N=2$ for $\Gamma/m=0.1$. For $\Gamma/m>0.1$, the number of oscillations decreases and tends rapidly to zero. Our results suggest that in real lattices, where dissipation
cannot be ignored, the existence of breathers is relevant only in the case of weak dissipation. Actually, this general regime, which should present unexpected features for some parameters range, remains to be explored carefully and will be discussed elsewhere.

**IV. STRONGLY DISSIPATIVE OR DIFFUSIVE REGIME**

Let us now focus on the overdamped or diffusive case \((m=0)\). Equation (2.4) then reduces to

\[
\Gamma \frac{dV_n}{dt} = k[V_{n+1} - 2V_n + V_{n-1}] \\
+ \left[ \Gamma \frac{\omega^2 + 2kV_n - \Gamma \omega}{\tau V_n^2 + 2k(\tau^2 - 1) V_n}{1 - V_n^2} \right].
\]

In order to compare our results (see hereafter) to those of Bressloff and Rowlands [11], we perform the following transformations. We divide the two members of (4.1) by \(\Gamma\) and set \(D=k/\Gamma\), \(\epsilon_D=\omega/(\tau D)\) and \(\alpha_0=2(1-\tau^2)\), to obtain

\[
\frac{dV_n}{dt} = D[V_{n+1} - 2V_n + V_{n-1}] \\
+ D\left[ \epsilon_D\left(1 - \alpha_0/2\right) - V_n^2 \right] - \frac{\alpha_0 V_n}{1 - V_n^2} + 2V_n.
\]

Equation (4.2) is a discrete reaction diffusion [7] equation of the form:

\[
\frac{dV_n}{dt} = D[V_{n+1} - 2V_n + V_{n-1}] + f(V_n),
\]

with kink-shaped solution (2.7) provided that, the potential \(V(V_n)\) is well chosen. Indeed, the potential corresponding to \(f(V_n)\) is

\[
V(V_n) = D\left[ \epsilon_D(1 - \alpha_0/2)V_n + V_n^2 \\
- \epsilon_D \frac{V_n^2}{3} + (\alpha_0/2)\ln(1 - V_n^2) \right].
\]

Note that, for \(D=1\), relation (4.3) is identical to the equation (1.3) of Bressloff and Rowlands in [11]. The potential (4.4), represented in Fig. 5 as a function of \(V_n\), is asymmetric with two nondegenerate minima \((a\) and \(b\) in Fig. 5). The value of \(\alpha_0\) is imposed by \(F\) and the velocity of the discrete front solution of (4.2) depends on the choice of \(\tau = \tanh(\alpha_0)\). The velocity of the front expressed in terms of the new parameters \(\alpha_0\) and \(\epsilon_D\) is

\[
c = \frac{\epsilon_D D}{\tau} = \frac{\epsilon_D D}{\sqrt{1 - \alpha_0^2}}.
\]
Substituting relations of \( \sim \) modify the coupling constant of this lattice without changing Schmidt’s model that we have introduced, interpolates between the conservative case for different values of the parameter \( \alpha \). As a result, in this case, we find that propagation failure occurs, as shown also by Bressloff and Rowlands. We have then observed that the collision between two wave fronts traveling in opposite directions results into their annihilation.

V. CONCLUDING REMARKS

The generalized dissipative nonlinear Klein-Gordon lattice model that we have introduced, interpolates between the Schmidt’s model [9] in the conservative regime and the reaction diffusion model of Bressloff and Rowlands [11] in the overdamped regime. Like Flach and coworkers [12] we have used an inverse method. However, contrary to these authors who have considered the conservative and dissipative lattice models separately, we have investigated a lattice model which includes both inertia and dissipation, like many systems in the real world. We have shown analytically that our lattice equations admit exact discrete kink or traveling wave fronts solutions if the double-well on-site potential is adequately chosen. In the nondissipative regime, we have checked numerically that discrete kink (antikink) solutions can propagate freely without experiencing any discreteness effects. Kink-antikink collisions reveal that static or moving discrete breathers with finite but with physically interesting lifetimes can be generated. In the general dissipative regime, discrete kinks can propagate freely and the lifetimes of the discrete breathers that can be created by collisions between these kinks depend on the importance of the dissipation. These results are interesting: they suggest that, for the real lattices where dissipation cannot be ignored, discrete kinks can travel and combine to generate discrete breathers with reasonable lifetimes, if dissipative effects are weak enough compared to inertial effects. In the overdamped or diffusive regime, discrete kinks or wave fronts can travel without experiencing any lattice effects or propagation failure. However, in this case, their collisions are totally inelastic. We would like to point out again that our model and results are relevant for physical systems in which the discreteness of the lattice is important. Obviously, further studies are necessary in the general dissipative regime to determine all the properties of these kinks with exceptional mobilities. In conclusion, we believe that the understanding of discrete nonlinear models is an active and attractive topic of the current research. Since realistic physical models are rather complicated, it is extremely important to develop the basic concepts with the help of simple lattice models with exact solutions.

APPENDIX: CALCULATIONS DETAILS

Starting from (2.1), we seek a total potential \( V(u_n) = F u_n + U(u_n) \) under the form (2.3) such that (2.2) is an exact solution of (2.1). Setting \( T_n = \tanh(\omega t - \kappa x) \) and \( \tau = \tanh(\kappa x) \) we then get

\[
\frac{d u_n}{d t} = u_0 (1 - T_n^2) \omega ,
\]

\[
\frac{d^2 u_n}{d t^2} = -2 u_0 \omega^2 (1 - T_n^2) T_n ,
\]

\[
\left[ \frac{dV(u_n)}{d u_n} \right] = F - A u_n + B_0 u_n^2 + B u_n^3 + C_0 u_n^4 + C u_n^5 + D_0 u_n^6 + D u_n^7 + \ldots .
\]

Substituting relations (A1), (A2), and (A3) in (2.1) yields

\[
-2 m u_0 \omega^2 T_n (1 - T_n^2) + \Gamma \omega u_0 (1 - T_n^2) - 2 u_0 k (1 - \tau^2) T_n (1 + \tau^2 T_n^2 + \tau^4 T_n^4 + \ldots ) + 2 k u_0 T_n + F - A u_0 T_n + B_0 u_0^2 T_n^2 + B u_0^3 T_n^3 + C_0 u_0^4 T_n^4 + C u_0^5 T_n^5 + D_0 u_0^6 T_n^6 + D u_0^7 T_n^7 + \ldots = 0.
\]
Writing this equation as a power series in $T_n$, we get

$$\begin{align*}
(F + \Gamma \omega u_0) + T_n^2(2ku_0 - Au_0 - 2mu_0 \omega^2 - 2ku_0(1 - \tau^2)) + T_n^3(B_0u_0^2 - \gamma \omega u_0) + T_n^3(2mu_0 \omega^2 - 2ku_0(1 - \tau^2) \tau^2 + Bu_0^3) \\
+ T_n^4(C_0u_0^3 + T_n[Cu_0 - 2ku_0(1 - \tau^2) \tau^4 + T_n^4(Du_0^5 - 2ku_0(1 - \tau^2) \tau^6)] + \cdots = 0.
\end{align*} \tag{A6}$$

Equation (A6) is satisfied for all $t$ and all $n$, if each coefficient of the series vanishes, yielding:

$$F = -\Gamma \omega u_0, \quad C_0 = 0, \quad A = 2(k \tau^2 - m \omega^2), \quad C = \frac{2}{u_0}[k \tau^4(1 - \tau^2)],$$

$$B_0 = \frac{\Gamma \omega}{u_0}, \quad D_0 = 0, \quad (A7)$$

$$B = \frac{2}{u_0}[k \tau^2(1 - \tau^2) - m \omega^2], \quad D = \frac{2}{u_0} k \tau^6 (1 - \tau^2).$$

The potential expression becomes

$$V(u_n) = k u_n^2 \left[ - \frac{\Gamma \omega}{k} \frac{u_n}{u_0} + \left( \frac{m \omega^2}{k} - 1 \right) \frac{u_n^2}{u_0} + \frac{\Gamma \omega}{3k} \frac{u_n^3}{u_0} - \frac{m \omega^2 u_n^4}{2k} \right]$$

$$+ \frac{1 - \tau^2}{\tau^2} \left( \frac{\tau^2 u_n^2}{2 u_0} + \frac{\tau^4 u_n^4}{3 u_0} + \frac{\tau^6 u_n^6}{4 u_0} + \cdots \right). \tag{A8}$$

Then (A8) can be rearranged to give

$$V(u_n) = k u_n^2 \left[ - \frac{\Gamma \omega}{k} \frac{u_n}{u_0} + \left( \frac{m \omega^2}{k} - 1 \right) \frac{u_n^2}{u_0} + \frac{\Gamma \omega}{3k} \frac{u_n^3}{u_0} - \frac{m \omega^2 u_n^4}{2k} \right]$$

$$- \frac{m \omega^2}{2k} \left( \frac{u_n^4}{u_0} + \left( \frac{1 - \tau^2}{\tau^2} \right) \ln \left( 1 - \tau^2 \frac{u_n^2}{u_0^2} \right) \right]. \tag{A9}$$

Equation (2.1) then becomes

$$m \frac{d^2 u_n}{dt^2} + \Gamma \frac{du_n}{dt} = k [u_{n+1} - 2u_n + u_{n-1}]$$

$$- k u_0 \left[ - \frac{\Gamma \omega}{k} + 2 \left( \frac{m \omega^2}{k} \right) \frac{u_n}{u_0} + 3 \frac{\Gamma \omega}{3k} \frac{u_n^2}{u_0^2} \right]$$

$$- 4 \frac{m \omega^2}{2k} \frac{u_n^3}{u_0^2} \left( \frac{1 - \tau}{\tau} \right) \frac{u_n^4}{u_0^4}. \tag{A10}$$

Finally, setting $v_n = \tau (u_n / u_0)$ and $\tau = \tanh(\kappa a)$, we obtain the general dimensionless equation of motion,

$$m \frac{d^2 v_n}{dt^2} + \Gamma \frac{dv_n}{dt} = k[ v_{n+1} - 2v_n + v_{n-1}] - k \frac{dV(v_n)}{dt} \tag{A11}$$

which becomes

$$m \frac{d^2 v_n}{dt^2} + \Gamma \frac{dv_n}{dt} = k[ v_{n+1} - 2v_n + v_{n-1}]$$

$$- k \left[ \alpha + 2\beta v_n + 3\gamma v_n^2 \right] + 4\delta v_n^3 + \xi v_n \left[ 1 - v_n \right]. \tag{A12}$$

with $\alpha = -\Gamma \omega / k$, $\beta = (m \omega^2 / k - 1)$, $\gamma = \Gamma \omega / (k \tau)$, $\delta = -m \omega^2 / (2k \tau^2)$, $\xi = -2(\tau^2 - 1)$.